February 9, 2008

EXACT DEFORMATIONS OF QUANTUM GROUPS; APPLICATIONS TO THE AFFINE CASE

C. Fronsdal

Department of Physics

University of California, Los Angeles, CA 90095-1547

Abstract. This paper continues our investigation of a class of generalized quantum groups. The "standard" R-matrix was shown to be the unique solution of a very simple, linear recursion relation and the classical limit was obtained in the case of quantized Kac-Moody algebras of finite type. Here the standard R-matrix for generalized quantum groups is first examined in the case of quantized affine Kac-Moody algebras. The classical limit yields the standard affine r-matrices of Belavin and Drinfeld. Then, turning to the general case, we study the exact deformations of the standard R-matrix and the associated Hopf algebras. They are described as a generalized twist, $R_\epsilon = (F^t)^{-1}RF$, where R is the standard R-matrix and F (a power series in the deformation parameter ϵ) is the solution of a linear recursion relation of the same type as that which determines R. Specializing again, to the case of quantized, affine Kac-Moody algebras, and taking the classical limit of these esoteric quantum groups, one re-discovers the esoteric affine r-matrices of Belavin and Drinfeld, including the elliptic ones. The formulas obtained here are easier to use than the original ones, and the structure of the space of classical r-matrices (for simple Lie algebras) is more tranparent. In addition, the r-matrices obtained here are more general in that they are defined on the central extension of the loop groups.

1. Introduction.

It is now known that most Lie bi-algebras can be quantized, but there is not yet a workable universal construction of the corresponding quantum groups. For simple Lie algebras, and for the affine Kac-Moody algebras, Belavin and Drinfeld [BD] gave a complete classification of the r-matrices; and thus, of the associated coboundary bi-algebras. They are the most interesting Lie bialgebras and their quantized versions is one of the subjects of this paper.

Our strategy is to construct the universal R-matrices for a class of generalized standard quantum groups (those that "commute with Cartan"), and to discover the others by means of deformation theory. In a previous report we have calculated all deformations of a certain type, up to first order in the deformation parameter. Here exact deformations are obtained, to all orders in the deformation parameter. Unfortunately, we still do not have a good characterization of a category within which the deformations should be sought; the type of deformations examined is thus somewhat *ad hoc* (see Eq.(1.14) below); nevertheless, all the trigonometric r-matrices are recovered in the classical limit, with their central extensions. In addition, the elliptic r-matrices turn up as a special case; these enigmatic objects thus find their natural place.

The standard R-matrix is constructed by a method that has already been put to some effect in a previous paper. (The same method was used by Lusztig [L] in a more special context.) The present work encompasses all the simple quantum groups (including the multiparameter or twisted versions constructed by Reshetikhin [R]) and the quantized Kac-Moody algebras, as well as other coboundary Hopf algebras that have nothing to do with Lie groups and that have not yet been investigated in detail. In this paper we specialize to the case of affine Kac-Moody algebras, except that Theorem 5.1 and Proposition 5.2 hold in the general case.

The affine Kac-Moody algebras are central extensions of loop algebras. The r-matrices obtained here are defined on the extended loop algebras and are therefore in this respect more general than those of Belavin and Drinfeld.

The standard R-matrix is a formal series

$$R = \exp\bigl(\varphi^{ab} H_a \otimes H_b\bigr)\Bigl(1 + \sum_{n=1}^{\infty} t_n\Bigr), \quad t_n = \sum t^{(\alpha')}_{(\alpha)}\, e_{-\alpha_1} \ldots e_{-\alpha_n} \otimes e_{\alpha'_1} \ldots e_{\alpha'_n}. \qquad (1.1)$$

Here $t^{(\alpha')}_{(\alpha)}$ are complex coefficients and $H_a, e_{\pm\alpha}$ are Chevalley-Drinfeld generators.

More precisely:

Definition. Let M, N be two countable sets, φ, H two maps,

$$\begin{aligned} \varphi: \quad & M \otimes M \to \mathbb{C} \ , \qquad a, b \to \varphi^{ab} \ , \\ H: \quad & M \otimes N \to \mathbb{C} \ , \qquad a, \beta \to H_a(\beta) \ . \end{aligned} \tag{1.2}$$

Let $\mathcal{A}$ or $\mathcal{A}(\varphi, H)$ be the universal, associative, unital algebra over $\mathbb{C}$ with generators $\{H_a\}\, a \in M,\ \{e_{\pm\alpha}\}\, \alpha \in N$, and relations

$$[H_a, H_b] = 0 \ , \quad [H_a, e_{\pm\beta}] = \pm H_a(\beta) e_{\pm\beta} \ , \tag{1.3}$$

$$[e_\alpha, e_{-\beta}] = \delta_\alpha^\beta \big(e^{\varphi(\alpha,\cdot)} - e^{-\varphi(\cdot,\alpha)}\big) \ , \tag{1.4}$$

with $\varphi(\alpha,\cdot) = \varphi^{ab} H_a(\alpha) H_b,\ \varphi(\cdot,\alpha) = \varphi^{ab} H_a H_b(\alpha)$ and $e^{\varphi(\alpha,\cdot)+\varphi(\cdot,\alpha)} \neq 1, \alpha \in N$. The free subalgebra generated by $\{e_\alpha\}\ \alpha \in N$ (resp. $\{e_{-\alpha}\}\ \alpha \in N$) will be denoted $\mathcal{A}^+$ (resp. $\mathcal{A}^-$). The subalgebra $\mathcal{A}^0$ generated by $\{H_a\}\, a \in M$ will be called the Cartan subalgebra.

The sum in (1.1) is over all $\alpha_i \in N$ and all permutations (α') of the set (α). We set

$$t_1 = \sum_{\alpha \in N} e_{-\alpha} \otimes e_\alpha \ . \tag{1.5}$$

If the parameters of $\mathcal{A}$ are in general position, then all the other coefficients are determined by the requirement that R satisfy the Yang-Baxter relation

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}. \tag{1.6}$$

For special values of the parameters the R-matrix is still uniquely determined by the Yang-Baxter relation, but now it has to be defined on a quotient $\mathcal{A}/I$ where $I \in \mathcal{A}$ is a suitable ideal [F]. Quantized Kac-Moody algebras are characterized by the property that, for each pair (α, β), there is a positive integer $k = k_{\alpha\beta}$ such that the following relation holds

$$e^{\varphi(\alpha,\beta)+\varphi(\beta,\alpha)+(k-1)\varphi(\alpha,\alpha)} = 1. \tag{1.7}$$

In this case the ideal I is generated by the Serre relations

$$0 = \sum_{m=0}^{k} Q_m^k \ (e_\alpha)^m e_\beta \, (e_\alpha)^{k-m} \ , \tag{1.8}$$

with coefficients

$$Q_m^k = (-)^m e^{m\varphi(\alpha,\beta)} \ q^{m(m-1)/2} \begin{pmatrix} k \\ m \end{pmatrix}_q \ , \quad q := e^{\varphi(\alpha,\alpha)}. \tag{1.9}$$

We suppose Card N and Card M finite and interpret $A = 1 - k$ as the generalized Cartan matrix of a Kac-Moody algebra; this allows us to extend the terminology of the classification of Kac-Moody algebras to quantized Kac-Moody algebras.

Let $\mathcal{A}'_{cl}$ be the algebra obtained from $\mathcal{A}'$ when the relations (1.4) are replaced by

$$[e_\alpha, e_{-\beta}] = \delta_\alpha^\beta \big(\varphi(\alpha, \cdot) + \varphi(\cdot, \alpha)\big). \tag{1.10}$$

If $\mathcal{A}'_{cl}$ is a Kac-Moody algebra of finite type, resp. affine type, then we may say that $\mathcal{A}'$ is a quantized Kac-Moody algebra of finite type, resp. affine type. But because $\mathcal{A}'$ cannot be recovered from $\mathcal{A}'_{cl}$ an autonomous definition is preferable.

Definition 1.2. Let $\mathcal{A}'$ be as above; that is, the quotient of an algebra $\mathcal{A}$ as per Definition 1.1, with parameters satisfying (1.7), by the ideal generated by the Serre relations (1.8). We shall say that $\mathcal{A}'$ is a quantized Kac-Moody algebra of finite type if (i) Card $M =$ Card $N = l < \infty$, and (ii) the (symmetrizable) generalized Cartan matrix

$$A = 1 - k, \quad k_{\alpha\beta} = \big(\varphi(\alpha, \beta) + \varphi(\beta, \alpha)\big)/\varphi(\alpha, \alpha) \tag{1.11}$$

is positive definite with $A_{\alpha\beta} \in \{0, -1, \ldots\},\ \alpha \neq \beta$. * We shall say that $\mathcal{A}'$ is a quantized Kac-Moody algebra of affine type if (i) Card $M = 1 +$ Card $N < \infty$, and (ii) the generalized Cartan matrix is positive semi-definite with $A_{\alpha\beta} \in \{0, -1, \ldots\},\ \alpha \neq \beta$ and all its principal minors are positive definite.

A quantized affine Kac-Moody algebra can be described as follows. Let $\hat{\mathcal{A}}'$ be as above, with parameters satisfying (1.7) and Serre relations (1.8), with root generators $\{e_{\pm\alpha}\}\, \alpha = 0, \ldots, l$ and Cartan generators $H_1, \ldots, H_l$, c, d, such that the subset that consists of $\{e_{\pm\alpha}\}\, \alpha \neq 0$ and $H_1, \ldots, H_l$ generates a subalgebra $\mathcal{A}'$ that is a quantized Kac-Moody algebra of finite type. Let $\hat{\varphi}$ refer to $\hat{\mathcal{A}}'$ and φ to $\mathcal{A}'$, and suppose that

$$\hat{\varphi} = \varphi + u\, c \otimes d + (1 - u)\, d \otimes c, \quad [d, e_{\pm\alpha}] = \pm\delta_\alpha^0\, e_{\pm 0}. \tag{1.12}$$

with some $u \in \mathbb{C}$. Suppose that c is central and that the extra root defined by $[H_a, e_0] = H_a(0) e_\alpha$ is such as to make the generalized Cartan matrix of $\hat{\mathcal{A}}'$ positive semi-definite with all its principal minors positive. Then $\hat{\mathcal{A}}'$ is a quantized affine Kac-Moody algebra.

---

* We are here assuming that the relation (1.7) is solved by the exponent taking the value zero.

The classical r-matrix associated with R is defined after a rescaling of the generators - Eq.(2.4) - as the coefficient of $\hbar$ in the expansion $R = 1 + \hbar r + o(\hbar^2)$; it satisfies the classical Yang-Baxter relation. Note that $r + r^t \neq 0$; the antisymmetric part of r satisfies the modified classical Yang-Baxter relation. We calculate this classical r-matrix, dealing separately with the following cases: First the unextended loop algebras, untwisted in Section 2, twisted in Section 3; then, in Section 4, the full Kac-Moody algebras.

Our second subject is the calculation of exact deformations of the standard R-matrix, satisfying the Yang-Baxter relation, in the wider context of the bialgebras $\mathcal{A}$ and $\mathcal{A}' = \mathcal{A}/I$ described above. We set

$$R_\epsilon = R + \epsilon R_1 + o(\epsilon^2),$$

and suppose that R_1 is driven by a term of the type

$$Se_{-\rho} \otimes e_\sigma + S' e_\sigma \otimes e_{-\rho}, \quad S, S' \in \mathcal{A}'^0. \tag{1.13}$$

Such deformations exist under certain conditions on the parameters; then S and S' and the remaining terms in R_ϵ (a formal power series in ϵ with constant term R) are determined by the Yang-Baxter relation. An exact formula (to all orders in ϵ) in closed form for R_ϵ is obtained for the case of elementary deformations, when R_1 is a single term of the type (1.13). In the general case of compound deformations, when (1.13) is replaced by a sum of terms of the same type, we obtain exact deformations in the form of a generalized twist.

Let R be the R-matrix of a coboundary Hopf algebra $\mathcal{A}'$, and $F \in \mathcal{A}' \otimes \mathcal{A}'$, invertible. Then

$$\tilde{R} := (F^t)^{-1} R F \tag{1.14}$$

satisfies the Yang-Baxter relation if F satisfies the following relation

$$\big((1 \otimes \Delta_{21})F\big)F_{12} = \big((\Delta_{13} \otimes 1)F\big)F_{31}. \tag{1.15}$$

(See Theorem 5.1 for the complete statement.) Though it is not quite germaine to our discussion, it may be worth while to point out that, if R is unitary, then so is $\tilde{R}$; the formula (1.14) therefore yields a large family of (mostly) new unitary R-matrices.

Applying this to our context, we find that the relation (1.15)is equivalent to a simple, linear recursion relation that can be reduced to the same form as the recursion relation

that determines the coefficients in the expansion of R. It has a unique solution that can be expressed directly in terms of the coefficients in (1.1). Just as in the standard case, this leads to a simple equation for the classical r-matrix, from which the latter is determined to all orders.

In Section 6 we specialize to the case of quantized, affine Kac-Moody algebras and take the classical limit, to recover the esoteric, affine r-matrices of the simple Lie algebras, with their central extensions. The result agrees with that of Belavin and Drinfeld, except that they did not include the central extension. The formulas obtained in this paper are more transparent and simpler to use.

Finally, in Section 7, we deal with a very special case, to discover that the elliptic r-matrices of $sl(N)$ form a special case among the deformed, trigonometric r-matrices. The universal R-matrix is expressed as an infinite product. It is shown, in the particular case of the elliptic R-matrix for $sl(2)$ in the fundamental representation, that this infinite product is both convergent and of practical utility; it reduces to the representation of elliptic functions in terms of infinite products, and the result is in perfect agreement with Baxter [B].

2. Untwisted loop algebras.

Consider a quantized affine Kac-Moody algebra $\hat{\mathcal{A}}'$, with generators $e_{\pm 0}, \ldots, e_{\pm l}$ and $H_1, \ldots, H_l, c, d$.

Definition 2. Positive root vectors are elements in $\mathcal{A}'^{+}$ defined recursively. (a) The generators e_α are positive root vectors. (b) If E_i, E_j are positive root vectors then so is

$$(1-x)^{-1}(E_i E_j - e^{\varphi(i,j)} E_j E_i), \quad x = e^{-\varphi(i,j)-\varphi(j,i)} \neq 1,$$

(c) All positive root vectors E_i are obtained in this way from the generators. Negative root vectors are in $\mathcal{A}'^{-}$ and are defined analogously.

It is easy to verify that

$$[E_i, E_{-i}] = e^{\varphi(i,\cdot)} - e^{\varphi(\cdot,i)}.$$

Let $\{E_i\}$ $i = 1, \ldots, n, +$ be the positive root vectors, labelled in such a way that

$$[e_\alpha, E_+] = 0 = [e_{-\alpha}, E_-], \tag{2.1}$$

and

$$[E_i, E_-] \in \mathcal{A}'^{0} \cdot \mathcal{A}'^{-}, \quad [e_{-\alpha}, E_+] \in \mathcal{A}'^{0} \cdot \mathcal{A}'^{+}. \tag{2.2}$$

Then we may refer to E_+ as a highest root vector.

Suppose that the extra root $H_a(0) = H_a(E_-)$, and pass to the associated untwisted loop algebra $\mathbb{C}[\lambda, \lambda^{-1}] \otimes \mathcal{A}'$ by substituting

$$\hat{\varphi} \to \varphi, \;\; e_0 := \lambda E_-, \quad e_{-0} := \lambda^{-1} E_+. \tag{2.3}$$

(Replacing $\hat{\varphi}$ by φ amounts to taking the quotient by the ideal generated by the central element c.)

The classical limit of R involves a parameter $\hbar$. We replace

$$x \to \sqrt{\hbar} x, \quad x = e_{\pm\alpha}, H_a. \tag{2.4}$$

and expand

$$R = 1 + \hbar r + o(\hbar^2).$$

After this, the t_n are all of order $\hbar$. Note that the truth of this last statement is not entirely trivial.

Now recall that the Yang-Baxter relation for R is equivalent to the recursion relation

$$[t_n, 1 \otimes e_{-\gamma}] = (e_{-\gamma} \otimes e^{\varphi(\gamma,\cdot)})t_{n-1} - t_{n-1}(e_{-\gamma} \otimes e^{-\varphi(\cdot,\gamma)}),\ n \geq 1. \tag{2.5}$$

To lowest order in $\hbar$ this becomes

$$\begin{aligned} &[t_1, 1 \otimes e_{-\gamma}] = e_{-\gamma} \otimes (\varphi + \varphi^t)(\gamma), \\ &[t_n, 1 \otimes e_{-\gamma}] = [e_{-\gamma} \otimes 1, t_{n-1}],\ \ n \geq 2, \end{aligned} \tag{2.6}$$

which is the same as

$$[1 \otimes e_{-\gamma} + e_{-\gamma} \otimes 1, r - \varphi] + [t_1, 1 \otimes e_{-\gamma}] = 0,\ \gamma = 0, \cdots, l, \tag{2.7}$$

with t_1 as in in (1.5), or

$$[1 \otimes e_{-\gamma} + e_{-\gamma} \otimes 1,\ r] = \varphi(\cdot, \gamma) \wedge e_{-\gamma}.$$

This result is just the classical limit of the relation $\Delta(e_{-\gamma})R = R\Delta'(e_{-\gamma})$, which explains why it determines r.

We normalize the root vectors so that the Casimir element takes the form

$$C = \varphi + \varphi^t + \sum E_{-i} \otimes E_i + \sum E_i \otimes E_{-i}. \tag{2.8}$$

Then

$$[e_{-\gamma}, E_{-i}] = cE_{-j} \text{ implies that } [E_j, e_{-\gamma}] = cE_i,\ \gamma \neq 0, \tag{2.9}$$

$$[e_{-0}, E_{-i}] = cE_j \text{ implies that } [E_{-j}, e_{-0}] = cE_i,\ \gamma \neq 0, \tag{2.10}$$

The classical r-matrix can be expressed as a formal power series in $x = \lambda/\mu$,

$$r = \varphi + \psi(x)^{ab} H_a \otimes H_b + \sum f_i(x) E_{-i} \otimes E_i + \sum g_i(x) E_i \otimes E_{-i}. \tag{2.11}$$

Now it is easy to work out the implications of Eq.(2.7), namely, first taking $\gamma \neq 0$,

$$\begin{aligned} 0 = [1 \otimes e_{-\gamma} + e_{-\gamma} \otimes 1, \psi(x)^{ab} H_a \otimes H_b + \sum f_i(x) E_{-i} \otimes E_i \\ + \sum g_i(x) E_i \otimes E_{-i}] + \sum e_{-\alpha} \otimes [e_\alpha, e_{-\gamma}] \\ = e_{-\gamma} \otimes \big(\psi(\gamma, \cdot) + (1 - f_\gamma)(\varphi + \varphi^t)(\gamma)\big) \\ + \big(\psi(\cdot, \gamma) - g_\gamma(\varphi + \varphi^t)(\gamma)\big) \otimes e_{-\gamma} \\ + \sum f_i [e_{-\gamma}, E_{-i}] \otimes E_i + \sum{}' f_i E_{-i} \otimes [e_{-\gamma}, E_i] \\ + \sum{}' g_i [e_{-\gamma}, E_i] \otimes E_{-i} + \sum g_i E_i \otimes [e_{-\gamma}, E_{-i}],\ \gamma \neq 0. \end{aligned} \tag{2.12}$$

The prime on $\sum'$ means that the summation is over roots that are not simple. Cancellation in the last two lines imply, in view of (2.9) and since the adjoint action is irreducible, that $f_i = f$, $g_i = g$, $i = 1, \dots, l$. Cancellation in the two first lines now tells us that $\psi \propto \varphi + \varphi^t$, hence ψ is symmetric, and it follows that $g = f - 1$. This gives us

$$r = \varphi + \sum E_{-i} \otimes E_i + g(x)C, \tag{2.13}$$

which is actually obvious: The two first terms is a special solution and the last term is the only thing that commutes with $\Delta_0(e_{-\gamma}) = 1 \otimes e_{-\gamma} + e_{-\gamma} \otimes 1$. Next, Eq.(2.7) with $\gamma = 0$,

$$\begin{aligned} 0 = [1 \otimes e_{-0} + e_{-0} \otimes 1,\ &\psi(x)^{ab} H_a \otimes H_b + \sum f_i(x) E_{-i} \otimes E_i \\ &+ \sum g_i(x) E_i \otimes E_{-i}] + \sum e_{-\alpha} \otimes [e_\alpha, e_{-0}] \\ = E_+ \otimes \big(&\frac{1}{\mu}\psi(0,\cdot) + (\frac{1}{\mu} - \frac{g}{\lambda})(\varphi + \varphi^t)(0)\big) \\ + \big(&\frac{1}{\lambda}\psi(\cdot,0) - \frac{f}{\mu}(\varphi + \varphi^t)(0)\big) \otimes E_+ \\ + &\frac{f}{\mu}\sum_{i \neq +}[E_+, E_{-i}] \otimes E_i + \frac{g}{\lambda}\sum_{i \neq +} E_i \otimes [E_+, E_{-i}]. \end{aligned} \tag{2.14}$$

This yields $g = xf$ and the result is that

$$r = \varphi + \sum E_{-i} \otimes E_i + \frac{x}{1-x}\, C, \quad x = \lambda/\mu, \tag{2.15}$$

which agrees with the simplest r-matrix in [BD], but in the notation of [J].

3. Twisted loop algebras.

The construction of a twisted affine Kac-Moody algebra [K] involves two simple Lie algebras, $\mathcal{L}$ and a subalgebra $\mathcal{L}_0$, such that $\mathcal{L}$ admits a diagram automorphism of order $k = 2$ or 3 to which is associated a Lie algebra automorphism μ that centralizes $\mathcal{L}_0$. The eigenvalues of μ are of the form ω^j, $j = 0, 1, \dots$, and $\mathcal{L} = \sum_{j=0}^{k-1} \mathcal{L}_j$, where $\mathcal{L}_j$ is the sum of the eigenspaces with eigenvalues $\omega^{j \bmod k}$. The restriction of the adjoint action of $\mathcal{L}$ to $\mathcal{L}_0$ acts irreducibly on each $\mathcal{L}_j$.

Now let $\{H_a, e_{\pm\alpha}\}\alpha = 1, \ldots n$ be a Chevalley basis for $\mathcal{L}_0$, and let E_+ be a highest weight vector (for the action of $\mathcal{L}_0$) in $\mathcal{L}_1$. Then $\{e_\alpha\}, E_-$ generate $\mathcal{L}$, and

$$[e_\alpha, E_+] = 0 = [e_{-\alpha}, E_-]. \tag{3.1}$$

The twisted loop algebra $\hat{\mathcal{L}} = \mathbb{C}[\lambda, \frac{1}{\lambda}] \otimes \mathcal{L}$ is generated by $\{e_{\pm\alpha}\}, \alpha = 0, \ldots, n$, with

$$e_0 = \lambda E_-, \quad e_{-0} = \frac{1}{\lambda} E_+. \tag{3.2}$$

This algebra is of the type $\mathcal{A}'_{cl}$, so our standard R-matrix applies. We define r in terms of the expansion of R in powers of $\hbar$ and work out the implications of the relations (2.7).

Let $\{E_i\}$ be a Weyl basis for $\mathcal{L}_0$ and normalize so that the Casimir element for that algebra is

$$C_0 = \varphi + \varphi^t + \sum E_{-i} \otimes E_i + \sum E_i \otimes E_{-i}. \tag{3.3}$$

Then a special solution of (2.7) with $\gamma \neq 0$ is given by the first two terms in (2.13) and the general solution is

$$r = \varphi + \sum E_{-i} \otimes E_i + \sum_0^{k-1} f_j C_j,$$

where C_j is the projection of the Casimir element C of $\mathcal{L}$ on $\mathcal{L}_j$, on the first factor. Now (2.7), with $\gamma = 0$:

$$\begin{aligned}
0 &= [1 \otimes e_{-0} + e_{-0} \otimes 1, \sum E_{-i} \otimes E_i + \sum f_j C_j] + \sum e_{-\alpha} \otimes [e_\alpha, e_{-0}] \\
&= \frac{1}{\mu} \sum [E_+, E_{-i}] \otimes E_i + \sum f_j \big(\frac{1}{\lambda} [1 \otimes E_+, C_j] + \frac{1}{\mu} [E_+ \otimes 1, C_j] \big) \\
&\qquad\qquad + \frac{1}{\mu} E_+ \otimes (\varphi + \varphi^t)(0) \\
&= \frac{1}{\mu} [E_+ \otimes 1, C_o] + \sum \big(\frac{f_j}{\lambda} [1 \otimes E_+, C_j] + \frac{f_{j-1}}{\mu} [E_+ \otimes 1, C_{j-1}].
\end{aligned} \tag{3.4}$$

This vanishes iff

$$f_1 = x(f_0 + 1), \;\; f_0 = x f_1, \;\; k = 2,$$

$$f_1 = x(f_0 + 1), \;\; f_2 = x f_1, \;\; f_0 = x f_2, \;\; k = 3,$$

That is,

$$f_j = \frac{x^j}{1 - x^k} C_j - \delta_j^0 \, C_0.$$

Finally, the unique solution is

$$r = \varphi + \sum E_{-i} \otimes E_i - C_0 + \frac{1}{1-x^k} \sum_0^{k-1} x^j C_j, \tag{3.5}$$

again in agreement with [BD], in the notation of [J].

Remark. Choose a basis of weight vectors in $\mathcal{L}_1$, then

$$C_1 = E_- \otimes E_+ + E_+ \otimes E_- + \ldots,$$

with unit coefficients for the contributions with highest weight. This follows from the normalization in (3.3) and fact that $1 \otimes E_+ + E_+ \otimes 1$ commutes with $C = \sum C_j$.

4. Including the central extension.

The untwisted case. The extension is recovered by omitting the replacement of $\hat{\varphi}$ by φ in (2.3). We can still represent the r-matrix as a power series in $x = \lambda/\mu$, but it is no longer true, as it was in the case of the loop group, that $[e_0, e_{-0}] = [E_-, E_+]$. Instead,

$$[e_0, e_{-0}] = (\hat{\varphi} + \hat{\varphi}^t)(0) = [E_-, E_+] + c. \tag{4.1}$$

More generally, for polynomials $f, g \in \mathbb{C}[\lambda, \frac{1}{\lambda}]$, and $x, y \in \mathcal{A}'_{cl}$,

$$[fx, gx] = fg[x, y] + c < x, y > \mathrm{Res}(f'g), \tag{4.2}$$

where the form $<,>$ is the invariant form on $\mathcal{A}'_{cl}$ normalized as follows: If the Casimir element is $C^{ij} x_i \otimes x_j$, then $< x_i, x_j >= (C^{-1})_{ij}$; $\mathrm{Res}(f)$ is the constant term in λf.

Remark. This normalization implies that

$$[fC_{12}, gC_{23}] = fg[C_{12}, C_{23}] + c_2 C_{13} \mathrm{Res}(f'g). \tag{4.3}$$

This change leaves (2.12) and (2.13) unaffected, while (2.14) becomes

$$\begin{aligned}0 = E_+ \otimes \big(\frac{1}{\mu}\psi(0,\cdot) + \frac{1}{\mu}(\hat{\varphi} + \hat{\varphi}^t)(0) + [e_0, g(x)E_-]\big) \\ + \big(\frac{1}{\lambda}\psi(\cdot,0) - \frac{f}{\mu}(\varphi + \varphi^t)(0)\big) \otimes E_+ \\ + \frac{f}{\mu}\sum_{i\neq +}[E_+, E_{-i}] \otimes E_i + \frac{g}{\lambda}\sum_{i\neq +} E_i \otimes [E_+, E_{-i}].\end{aligned}$$

The modification in the second term (φ replaced by $\hat{\varphi}$) is exactly compensated by a new contribution from the linear λ-term in g. (There is no linear μ-term in f.) The conclusion is that the new r-matrix is

$$\hat{r} = \hat{\varphi} + \sum E_{-i} \otimes E_i + \frac{x}{1-x} C. \tag{4.4}$$

<u>The twisted case.</u> It is easy to verify, with the help of the remark at the end of Section 3, that the restitution $\varphi \to \hat{\varphi}$ can be made without affecting the cancellations; so the result is that

$$\hat{r} = \hat{\varphi} + \sum E_{-i} \otimes E_i - C_0 + \frac{1}{1-x^k}\sum x^j C_j. \tag{4.5}$$

It is amusing to verify directly that the classical Yang Baxter relation for r,

$$YB(r) := [r_{12}, r_{13} + r_{23}] + [r_{13}, r_{23}] = 0,$$

implies the same relation for $\hat{r}$: The inclusion of the extra term in $\hat{\varphi}$ means that

$$YB(\hat{r}) = YB(r) + [r_{13}, (c \otimes d)_{23}]. \tag{4.6}$$

The evaluation of $YB(r)$ now has to take into account the new term (involving c) in Eq.(4.2). Actually, only $[r_{12}, r_{23}]$ is affected, and with the aid of Eq.(4.3) one finds that the new contribution is

$$YB(r) = c_2 \lambda \frac{d}{d\lambda} r_{13},$$

which exactly cancels the other term. In the twisted case one must use the following generalization of Eq.(4.3):

$$[f C_{j\,12}, g C_{j'\,23}] = fg[C_{j\,12}, C_{j'\,23}] + \delta_j^{j'} c_2\, C_{j\,13} \mathrm{Res}(f'g). \tag{4.7}$$

5. Deformations.

A deformation of the standard R-matrix is a formal series

$$R_\epsilon = R + \epsilon R_1 + \epsilon^2 R_2 + \dots\ . \tag{5.1}$$

Here R is given by Eq. (1.1), with the coefficients determined by the Yang-Baxter relation, and we determine $R_1, R_2, \dots$ so that R_ϵ will satisfy the same relation to each order in ϵ.

Recall that R is driven by the linear term; that is, by virtue of the Yang-Baxter relation, R is completely determined by the term t_1. It is natural to study deformations that are driven by a similar term, with fixed but non-zero weight:

$$R_1 = S(e_{\pm\sigma} \otimes e_{\pm\rho}), \tag{5.2}$$

with σ, ρ fixed and the factor S is in $\mathcal{A}'^0$.

The following result was obtained. Among the possibilities in (5.2) the only one that turns out to lead to a general class of deformations is

$$S(e_\sigma \otimes e_{-\rho}) + S'(e_{-\rho} \otimes e_\sigma), \;\; S \in \mathcal{A}^0. \tag{5.3}$$

Proposition 5.1. [F] Suppose that $R + \epsilon R_1$ is a first order deformation, satisfying the Yang-Baxter relation to first order in ϵ. Suppose also that the simplest term in R_1 has the form (5.3), with $S \neq 0$, then the parameters satisfy

$$e^{\varphi(\cdot,\rho) + \varphi(\sigma,\cdot)} = 1\ . \tag{5.4}$$

In this case R_1 is uniquely determined and has the expression

$$R_1 = (Ke_{-\rho} \otimes Ke_\sigma)R - R(Ke_\sigma \otimes Ke_{-\rho})\ , \tag{5.5}$$

with $K_\rho \;:=\; e^{\varphi(\cdot,\rho)} = K^\sigma \;\; := e^{-\varphi(\sigma,\cdot)}$.

Deformations of this type, involving a single pair (ρ, σ) for which (5.4) holds, is called an elementary deformation. To first order in ϵ, the problem being then linear, one obtains a more general space of deformations by adding the contributions of several such pairs,

$$R_1 = \sum_{(\sigma,\rho) \in [\tau]} \big(f_{-\rho} \otimes f_\sigma R - R f_\sigma \otimes f_{-\rho}\big). \tag{5.6}$$

Here the sum is over a subset $[\tau]$ of the pairs (σ, ρ); $\sigma \in \hat\Gamma_1$, $\rho \in \hat\Gamma_2$, where $\hat\Gamma_{1,2}$ are subsets of the set of positive generators, and

$$e^{\varphi(\sigma,\cdot)+\varphi(\cdot,\rho)} = 1, \quad (\sigma,\rho) \in [\tau]. \tag{5.7}$$

But not all such compounded, first order deformations are approximations to exact deformations (deformations to all orders in ϵ).

The deformed co-product was also calculated to first order in ϵ, and the result suggests an approach to the exact deformations. For the following result $\mathcal{A}'$ is any coboundary Hopf algebra.

Theorem 5.1. Let R be the R-matrix, Δ the coproduct, of a coboundary Hopf algebra $\mathcal{A}'$, and $F \in \mathcal{A}' \otimes \mathcal{A}'$, invertible, such that

$$\big((1\otimes\Delta_{21})F\big)F_{12} = \big((\Delta_{13}\otimes 1)F\big)F_{31}. \tag{5.8}$$

Then

$$\tilde R := (F^t)^{-1} R F \tag{5.9}$$

(a) satisfies the Yang-Baxter relation and (b) defines a Hopf algebra $\tilde{\mathcal{A}}$ with the same product and with co-product

$$\tilde\Delta = (F^t)^{-1}\Delta F^t. \tag{5.10}$$

Proof. (a) We substitute (5.8) into the expression $\tilde R_{12}\tilde R_{13}\tilde R_{23}$. Then use (5.7) to express $F_{12}(F_{31})^{-1}$ in terms of the co-products, and the intertwining property of R ($\Delta R = R\Delta'$) to shift the latter to the ends. The rest is obvious. (b) It is clear that $\tilde\Delta$ is an algebra homomorphism. We shall show that the twisted coproduct defined by $\tilde\Delta$ is co-associative:

$$\begin{aligned}(1\otimes\tilde\Delta_{23})\tilde\Delta(x) &= F_{32}^{-1}(1\otimes\Delta_{23})\tilde\Delta(x)F_{32}\\ &= F_{32}^{-1}(1\otimes\Delta_{23}\,(F^t)^{-1})(1\otimes\Delta_{23}\,\Delta(x))(1\otimes\Delta_{23}\,F^t)F_{32},\\ (\tilde\Delta_{12}+1)\tilde\Delta(x) &= F_{21}^{-1}(\Delta_{12}\otimes 1\,(F^t)^{-1})(\Delta_{12}\otimes 1\,\Delta(x))(\Delta_{12}\otimes 1\,F^t)F_{21}.\end{aligned}$$

Comparing the factors at either end one gets, in view of the co-associativity of Δ, a relation that reduces to (5.8) after re-numbering the spaces. The theorem is proved. *

* The connection between Eq.(5.8) and co-associativity was pointed out to me by Kajiwara.

The naturality of this construction is indicated by the following simple result.

Proposition 5.2. Let R, F be as in Theorem 5, and suppose that there is a second twist $\tilde{F} \in \tilde{\mathcal{A}} \otimes \tilde{\mathcal{A}}$, invertible, such that

$$\big((1 \otimes \tilde{\Delta}_{21})\tilde{F}\big)\tilde{F}_{12} = \big((\tilde{\Delta}_{13} \otimes 1)\tilde{F}\big)\tilde{F}_{31}.$$

Then the two twists can be combined to $G = F\tilde{F}$ satisfying

$$\big((1 \otimes \Delta_{21})G\big)G_{12} = \big((\Delta_{13} \otimes 1)G\big)G_{31}.$$

We return to our subject, with R again denoting the standard R-matrix of the algebra $\mathcal{A}' = \mathcal{A}/I$. We show first that interesting solutions of (5.8) exist. Then we do some preliminary calculations that help us make a general ansatz for F in the form of a double expansion, $F = \sum \epsilon^{nm} F_n^m$, and finally we derive a recursion relation for F_n^m that will allow us to calculate the classical limit.

Examples. An exact deformation of R, with first order term R_1 as in (5.5) is given by

$$F = e_q^{-\epsilon f_\sigma \otimes f_{-\rho}}, \tag{5.11}$$

with

$$f_\sigma := e^{-\varphi(\sigma,\cdot)} e_\sigma, \;\; f_{-\rho} := e_{-\rho} e^{\varphi(\cdot,\rho)} \tag{5.12}$$

The q-exponential is as follows: $q = e^{\varphi(\sigma,\rho)}$, ${e_q}^A := \sum A^n/[n!]_q$, $[n!]_q = [1!]_q \dots [n]_q$, $[n]_q = (q^n - 1)/(q-1)$. Note that, if $AB = qBA$, then $e_q^A e_q^B = e_q^{(A+B)}$. This construction works easily for some compound deformations, with (5.5) replaced by (5.6). Proposition 5.2 shows that an elementary twist F, of the simple form (5.11), can be combined with another elementary twist $\tilde{F}$, of the same type but with (σ, ρ) replaced by (σ', ρ'), only if $\tilde{\Delta}(f'_\sigma), \tilde{\Delta}(\rho')$ reduce to $\Delta(f'_\sigma), \Delta(\rho')$; that is, only when the four generators quommute among themselves.

Notation. From now on it will be convenient to use the generators $f_{\pm\alpha}$ defined in (5.12). The standard co-product then takes the form

$$\Delta f_\sigma = K^\sigma \otimes f_\sigma + f_\sigma \otimes 1, \quad \Delta f_{-\rho} = 1 \otimes f_{-\rho} + f_{-\rho} \otimes K_\rho,$$

with

$$K_\rho := e^{\varphi(\cdot,\rho)},\ \ K^\sigma := e^{-\varphi(\sigma,\cdot)}.$$

The general case of compound deformations is much more complicated. The calculations are manageable only so long as F can be constructed from elements of the type $f_\sigma \otimes f_{-\rho}$ only, with the factors in this order. A general result is Theorem 5.2 below. We need some preparation.

Proposition 5.3. Let R_ϵ be an exact deformation of the type

$$\begin{aligned} &R_\epsilon = (F^t)^{-1} R F, \qquad F = \sum \epsilon^n (F_n + \ldots), \\ &F_n = \sum_{(\sigma,\rho)\in[\tau]} F^{(\rho')}_{(\sigma)} f_{\sigma_1} \ldots f_{\sigma_n} \otimes f_{-\rho'_1} \ldots f_{-\rho'_n}, \end{aligned} \tag{5.13}$$

where $+\ldots$ stands for terms with less than n factors. Let Γ_1, Γ_2 be the subalgebras of $\mathcal{A}'^+$ generated by $\hat\Gamma_1, \hat\Gamma_2$. Then we have: (a) There is an isomorphism $\tau : \Gamma_1 \to \Gamma_2$, such that the set $[\tau]$ is the restriction of the graph of τ to $\hat\Gamma_1, \hat\Gamma_2$,

$$[\tau] = \{\sigma, \rho \,|\, \sigma \in \hat\Gamma_1,\, \rho = \tau\sigma \in \hat\Gamma_2\}. \tag{5.14}$$

(b) The elements F_n satisfy the recursion relations

$$[F_n, f_{-\sigma} \otimes 1] = (K^\sigma \otimes f_{-\rho}) F_{n-1} - F_{n-1}(K_\sigma \otimes f_{-\rho}), \quad (\sigma,\rho) \in [\tau], \tag{5.15}$$

as well as

$$[1 \otimes f_\rho, F_n] = F_{n-1}\big(f_\sigma \otimes K^\rho\big) - \big(f_\sigma \otimes K_\rho\big) F_{n-1}. \tag{5.16}$$

(c) These recursion relations have the unique solution

$$F^{(\rho')}_{(\sigma)} = -\bar t^{(\sigma')}_{(\sigma)}, \ \ (\rho'_1, \ldots, \rho'_n) = \tau(\sigma'_1, \ldots, \sigma'_n), \tag{5.17}$$

where the coefficients on the right are the same as in Eq.(1.1), except that φ is replaced by $-\varphi :\ t(\varphi) = \bar t(-\varphi)$.

Proof . We begin by offering some justification for the assumptions. In view of the form of R_1 it is expected that R_n is a sum of products of factors of three types:

$$e_{-\alpha} \otimes e_\alpha,\ \ f_{-\rho} \otimes f_\sigma,\ \ f_\sigma \otimes f_{-\rho}, \quad \sigma \in \hat\Gamma_1,\, \rho \in \hat\Gamma_2, \tag{5.18}$$

with coefficients in $\mathcal{A}'\otimes\mathcal{A}'$. In R_n , we isolate the terms with the highest number of factors of the third type,

$$X_n = \sum A^{(\rho')}_{(\sigma)}\big(e_{-\alpha_1}\dots e_{-\alpha_k}\otimes e_{\alpha_1}\dots e_{\alpha_k}\big)B^{(\rho')}_{(\sigma)}\big(f_{\sigma_1}\dots f_{\sigma_n}\otimes f_{-\rho'_1}\dots f_{-\rho'_n}\big),$$

We shall show that $R_n$contains $X_n\neq 0$.

Let

$$YB_\epsilon := R_{\epsilon\,12}R_{\epsilon\,13}R_{\epsilon\,23} - R_{\epsilon\,23}R_{\epsilon\,13}R_{\epsilon\,12}\in\mathcal{A}'\otimes\mathcal{A}'\otimes\mathcal{A}'. \tag{5.19}$$

All terms in YB_ϵ of order ϵ^n, that have n factors of the third type in spaces 1,2 are contained in

$$P_n := F_{n\,12}R_{13}R_{23} - R_{23}R_{13}F_{n\,12}. \tag{5.20}$$

For these terms to cancel among themselves X_n must take the form

$$X_n = RF_n,\quad F_n = F^{(\rho')}_{(\sigma)}\big(f_{\sigma_1}\dots f_{\sigma_n}\otimes f_{-\rho'_1}\dots f_{-\rho'_n}\big). \tag{5.21}$$

The sum is over all pairs $(\sigma,\rho)\in[\tau]$ and all permutations (ρ') of (ρ).

Next, the recursion relation (5.15) follows easily from the Yang-Baxter relation (more precisely from an examination of terms of low order in space 2), and (5.16) from a similar calculation. (Recursion relations of this type were examined in detail in [F], so we skip the details.)

We have $F_0=1$ and $F_1=\sum f_\sigma\otimes f_{-\rho}$. Taking $n=1$ in (5.15) or (5.16) one gets,

$$[f_\alpha, f_{-\beta}] = \delta^\beta_\alpha\big(e^{\varphi(\cdot,\alpha)} - e^{-\varphi(\alpha,\cdot)}\big), \tag{5.23}$$

which is confirmed by the definitions in (5.12) and the relation (1.4). When (5.15) is reduced to a recursion relation for the coefficients, then it turns out to agree (up to the sign of φ) with the recursion relations for the coefficients $t^{(\sigma')}_{(\sigma)}$ that is implied by (1.1) and (2.5). The integrability of these relations is precisely the statement (a) of the theorem, as follows easily from the analysis of these recursion relations in [F]. Finally, when (a) holds, then the relation (5.16) is equivalent to (5.15). The proposition is proved.

After these preliminary explorations we are able to formulate a general result.

Theorem 5.2. Let Γ_1, Γ_2 be subalgebras of $\mathcal{A}'^{+}$, generated by subsets $\hat{\Gamma}_1, \hat{\Gamma}_2$ of the generators, and $\tau : \Gamma_1 \to \Gamma_2$ an algebra isomorphism. Let $F \in \mathcal{A}' \otimes \mathcal{A}'$ be a formal series of the form

$$F = 1 + \sum_{n=1}^{\infty} \epsilon^n F_n, \quad F_n = \sum F_{(\sigma)}^{(\rho)} f_{\sigma_1} \dots f_{\sigma_n} \otimes f_{-\rho_1} \dots f_{-\rho_n}. \tag{5.24}$$

The second sum is here over all $\sigma_i \in \hat{\Gamma}_1, \rho_i \in \hat{\Gamma}_2$. (!) Note that F_n is a power series in ϵ. Suppose that F satisfies (5.8), then

$$F_1 = -\sum_{\tau^m \sigma = \rho} \epsilon^{m-1} (f_\sigma \otimes f_{-\rho}), \tag{5.25}$$

and

$$(1 \otimes K_\rho \partial_\rho) F_n + \sum_{\tau^m \sigma = \rho} \epsilon^m \, [1 \otimes f_\sigma, F_n] + \sum_{\tau^m \sigma = \rho} \epsilon^{m-1} (f_\sigma \otimes K^\sigma) F_{n-1} = 0. \tag{5.26}$$

This last relation is satisfied for $n = 1$ by (5.25), with $F_0 = 1$. With F_1 thus fixed, $F_2, F_3, \dots$ are determined recursively and uniquely.

Notation. The sums in (5.25-6), and similar sums to follow, should be understood to run over $\sigma \in \hat{\Gamma}_1$ and over all values of the integer m such that $\tau^m \sigma$ is defined; that is, all values of m such that $\tau^{m-1} \sigma \in \hat{\Gamma}_1$.

Proof. That Eq.(5.8) implies (5.25) and (5.26) is a simple calculation; one collects all terms that have exactly one generator in the second space. Let us verify that the recursion relation is satisfied for $n = 1$ by (5.25). The second term is

$$-\sum_{\tau^{m'} \sigma' = \rho'} \epsilon^{m'} \sum_{\tau^m \sigma = \rho} \epsilon^{m-1} f_{\sigma'} \otimes [f_\sigma, f_{-\rho'}].$$

The commutator is

$$[f_\sigma, f_{-\rho'}] = e^{\varphi(\cdot, \rho')} - e^{-\varphi(\rho', \cdot)} = e^{\varphi(\cdot, \tau^{m'} \sigma')} - e^{\varphi(\cdot, \tau^{m'+1} \sigma')}.$$

The double sum reduces to $\sum_{\tau^m \sigma = \rho} \epsilon^{m-1} f_\sigma \otimes (K^\sigma - K_\rho)$ and (5.26) reduces to an identity. It remains to prove that (5.26) has a unique solution. Consider first the case that $\hat{\Gamma}_1 \cap \hat{\Gamma}_2$

is empty; then the second term in (5.26) vanishes and the third term reduces to the term $m = 1$. The recursion relation then reduces to the same form as that which determines the coefficients of the standard R-matrix, which is known to be integrable [F]. (In this case Proposition 5.3 is the complete solution of the problem, for there are no terms "$+\dots$" in (5.13).) In the general case, when $\hat\Gamma_1 \cap \hat\Gamma_2$ can be non-empty, the second term in (5.26) makes the solution more difficult, but the existence of a solution can still be proved. To do this we expand F_n as a power series in ϵ, with constant term

$$F_n^1 = \sum F_{(\sigma)}^{(\tau\sigma')} f_{\sigma_1'} \dots f_{\sigma_n'} \otimes f_{-\tau\sigma_1'} \dots f_{-\tau\sigma_n'},$$

and determine the coefficients recursively. The problem is therefore always the integrability of $K_\rho \partial_\rho X = Y, \rho \in \hat\Gamma_2$, with $Y \in \mathcal{A}'$ given, and this is known [F] to have a unique solution in $\mathcal{A}'$, as already noted. The theorem is proved.

The converse, that the solution of (5.26) with $F_0 = 1$ and F_1 given by (5.25) satisfies (5.8) (and therefore gives a solution of the Yang-Baxter relation) was proved only in the special case that $\hat\Gamma_1 \cap \hat\Gamma_2$ is empty. Note, however, that the exact form (5.25) of F_1 can be inferred directly from the Yang-Baxter relation. Further direct computation supports the idea that R_ϵ always has the form $(F^t)^{-1} R F^t$, with F of the form assumed in (5.24). This is strong support for the belief that the solution of the recursion relation (5.26), which was proved to exist always, actually furnishes the solution to the problem of exact deformations in the general case. As we shall see, additional favorable evidence comes from an examination of the classical limit. To prepare for this we need

Proposition 5.4. Let

$$F_n^m = \sum_{\rho = \tau^m \sigma} \bar t_{(\sigma)}^{(\sigma')} f_{\sigma_1} \dots f_{\sigma_n} \otimes f_{-\tau^m\sigma_1'} \dots f_{-\tau^m\sigma_n'}, \quad F_0^m = 1, \tag{5.27}$$

in which the sum extends over $\sigma_i \in \hat\Gamma_1$, (σ') a permutation of (σ), and the coefficients $\bar t_{(\sigma)}^{(\sigma')}$ are the same as in (5.17). Then the unique solution of (5.26) is

$$\begin{aligned} F_n &= \sum_{\Sigma n_i = n} \epsilon^{n_2 + 2n_3 + \dots} F_{n_1}^1 F_{n_2}^2 \dots \; = F_n^1 - \epsilon F_{n-1}^1 F_1^2 + \epsilon^2 \left(F_{n-2}^1 F_2^2 + F_{n-1} F_1^3 \right) + \dots, \\ F &= \sum \epsilon^n F_n = \sum \epsilon^{n_1 + 2n_2 + \dots} F_{n_1}^1 F_{n_2}^2 \dots = F^1 F^2 \dots, \quad F^m = \sum \epsilon^{nm} F_n^m. \end{aligned} \tag{5.28}$$

6. Esoteric r-matrices.

We specialize to the case of a quantized Kac-Moody algebra of finite type.

Proposition 6.1. If $\mathcal{A}'$ is a quantized Kac-algebra of finite type, then $\hat{\Gamma}_1$ is a proper subset of the set of positive generators and $\tau^{m+1}\hat{\Gamma}_1 \cap \hat{\Gamma}_1$ is a proper subset of $\tau^m\hat{\Gamma}_1 \cap \hat{\Gamma}_1$.

Proof. Suppose that the statement is false. Then there is $f_\sigma \in \hat{\Gamma}_1$ such that $\tau^m f_\sigma \in \hat{\Gamma}_1$ for all m, and consequently $\tau^k f_\sigma = f_\sigma$ for some k. But the condition (5.7), in the classical limit, implies that

$$\varphi(\tau^m\sigma, \cdot) + \varphi(\cdot . \tau^{m+1}\sigma) = 0.$$

Summing over $m = 0, 1, \ldots, k-1$ we obtain

$$\sum_m (\varphi + \varphi^t)(\tau^m\sigma) = 0,$$

which contradicts the fact that the Killing form is non-degenerate.

In the classical limit

$$R_\epsilon = 1 + \hbar r_\epsilon + o(\hbar^2) \quad r_\epsilon = r + \epsilon + o(\epsilon^2). \tag{6.1}$$

In the case of an exact elementary deformation R_ϵ, the associated exact deformation r_ϵ of r coincides with the first order,

$$r_\epsilon = r + \epsilon r_1. \tag{6.2}$$

Consider the general case of an exact deformation of R of the form postulated in Theorem 5.2. Define X_ϵ by

$$F = 1 + \hbar X_\epsilon + o(\hbar^2), \tag{6.3}$$

so that

$$r_\epsilon = r + X_\epsilon - X_\epsilon^t. \tag{6.4}$$

Notation. In this section the symbols $\Gamma_{1,2}$ stand for Lie algebras, the classical limits of the algebras so designated until now.

From the fact that the coefficients in (5.27) are the same as the coefficients in (1.1), and the known classical limit of the standard R-matrix for a Kac-Moody algebra of finite type, we get without calculation that

$$X_\epsilon = -\sum_m \sum_{E_i \in \Gamma_1} \epsilon^{nm} E_i \otimes E_{-\tau^m i}, \tag{6.5}$$

in which n is the height of E_i. The normalization is the same as in Sections 2-4; more precisely it is fixed as follows. (a) The set $\{E_i\}$ includes the generators of Γ_1. (b) The statement (2.9). * Consequently,

$$r_\epsilon = r - \sum_m \sum_{\substack{E_i \in \Gamma_1 \\ E_j = \tau^m E_i}} \epsilon^{nm} \; E_i \wedge E_{-j}. \tag{6.6}$$

The sums are finite, by Proposition 6.1. A renormalization exists that reduces the numerical coefficients to unity (ϵ now interpreted as in $\mathbb{C}$); the result is in complete agreement with [BD].

Deformations in the affine case. Let $\mathcal{A}'$ be a quantized Kac-Moody algebra of affine type. Two cases should be distinguished. If the subsets $\hat{\Gamma}_{1,2}$ of positive roots do no include the imaginary root e_0, then the formula (6.6) applies without change, except that now r is one of the standard affine r-matrices determined earlier, Eq.s (2.15), (3.5), (4.4) or (4.5). There is nothing more to be said about this case and we turn our attention to the other one.

What merits special attention is the possibility that the first order deformation (5.6) may include one of the following

$$e_0 \wedge e_{-\rho} = \mu(E_- \otimes e_{-\rho}) - \lambda(e_{-\rho} \otimes E_-), \tag{6.7}$$

* Condition (b) can be re-phrased as follows. Let Γ_1^- be the Lie algebra generated by $\{f_{-\sigma}\}$, $f_\sigma \in \hat{\Gamma}_1$ and Γ the Lie algebra generated by $\{f_{\pm\sigma}\}$, $f_\sigma \in \hat{\Gamma}_1$. Then

$$\sum_{E_i \in \Gamma_1} E_i \otimes E_{-i}$$

is the projection on $\Gamma_1 \otimes \Gamma^-$ of a Γ-invariant element of $\Gamma \otimes \Gamma$.

or

$$e_\sigma \wedge e_{-0} = \lambda^{-1}(e_\sigma \otimes E_+) - \mu^{-1}(E_+ \otimes e_\sigma), \tag{6.8}$$

with

$$\varphi(\cdot,\rho) + \varphi(0,\cdot) = 0, \quad \text{resp.} \quad \varphi(\cdot,0) + \varphi(\sigma,\cdot) = 0, \tag{6.9}$$

which implies that $\rho \neq 0$, resp. $\sigma \neq 0$. A simple renormalization, that connects the principal picture to the homogeneous picture, brings (6.8) to the form

$$e_\sigma \wedge e_{-0} = \sqrt{\mu/\lambda}(e_\sigma \otimes E_+) - \sqrt{\lambda/\mu}(E_+ \otimes e_\sigma).$$

To deal with the general case of exact deformations it is useful to note the following

Proposition 6.2 If $\mathcal{A}'$ is a quantized Kac-Moody of affine type, then *either* the statement about $\hat{\Gamma}_1$ in Proposition 6.1 continues to hold, *or* $\mathcal{A}'$ is of type $A^{(1)}_{N-1}$, $\hat{\Gamma}_1$ consists of all the positive generators, and τ generates the cyclic group of order N.

Proof. Suppose there is $f_\sigma \in \hat{\Gamma}_1$ such that $\tau^N f_\sigma = f_\sigma$ for some N. Then the Killing form is degenerate. But it is known [K] that any subalgebra of a Kac-Moody algebra of affine type, obtained by removing one generator, is a Kac-Moody algebra of finite type. It follows that $\hat{\Gamma}_1$ contains all the positive generators and exactly one τ orbit. Then $\hat{\Gamma}_1 = \hat{\Gamma}_2$ and τ lifts to an isomorphism of the Dynkin diagram, which implies the result.

In this section we exclude the exceptional case. This means that the classical limit of Γ_1 is a finite dimensional Lie algebra, so that (6.6) can be applied directly, since the sum is finite.

Alternatively, the classical limit of an exact deformation can be found with the help of the recursion relation

$$(1 + K_\rho \partial_\rho) F^m_n = -(f_\sigma \otimes K_\rho) F^m_{n-1}, \quad \tau^m \sigma = \rho, \tag{6.10}$$

or better, the equivalent relation

$$[1 \otimes f_\rho, F^m_n] = -\Big((f_\sigma \otimes K_\rho) F^m_{n-1} - F^m_{n-1}(f_\sigma \otimes K^\rho)\Big) \tag{6.11}$$

for $F_n^m = \delta_n^0 + \hbar X_n^m + o(\hbar^2)$. This implies that $X^m = \sum_{n=0,1,\dots} \epsilon^{mn} X_n^m$ (a finite sum) is the unique solution (of the form that appears in (6.5)) of

$$[1 \otimes f_\rho + \epsilon^m f_\sigma \otimes 1, X^m] = \epsilon^m f_\sigma \otimes (\varphi + \varphi^t)(\rho), \quad \tau^m \sigma = \rho \in \hat{\Gamma}_2. \tag{6.12}$$

Example. Let $\mathcal{A}'_{cl}$ be the untwisted, affine Kac-Moody algebra $\tilde{\mathcal{L}}, \mathcal{L} = sl(N)$. A set of positive Serre generators is provided by the unit matrices $e_i = e_{i,i+1},\ i = 1, \dots N-1$. Set $e_N = e_0 = \lambda e_{N1}$. The "most esoteric" deformation (the one with the largest Γ_1) is defined as follows. Take Γ_1 to be generated by $e_i,\ i = 1, \dots N-1$, and $\tau e_i = e_{i+1},\ i = 1, \dots N-1$. Then $X^m = \sum_n \epsilon^{nm} X_n^m$ with

$$X_n^m = -\sum_{i+m+n \le N} e_{i,i+n} \otimes e_{i+m+n,i+m} - \sum_{i+m+n=N+1} e_{i,i+n} \otimes \lambda^{-1} e_{1,i+m}$$

and

$$r_\epsilon = r + \Big(\sum \epsilon^{nm} X_n^m - \text{transpose}\Big).$$

Taking $N = 3$ one obtains

$$r_\epsilon = r - \Big(\epsilon\, e_{12} \otimes e_{32} + \epsilon^2 e_{13} \otimes \lambda^{-1} e_{12} + \epsilon^2 e_{12} \otimes \lambda^{-1} e_{13} - \text{transpose}\Big),$$

and the renormalization $e_{ij} \to \lambda^{\frac{j-i}{3}} e_{ij}$ gives the final result

$$\begin{aligned} r_\epsilon = r &- \epsilon\{\xi^{-1} e_{12} \otimes e_{32} + \xi^{-1} e_{23} \otimes e_{13} + \xi^{-2} e_{13} \otimes e_{12}\} - \epsilon^2 \xi^{-1} e_{12} \otimes e_{13} \\ &+ \epsilon\{\xi e_{32} \otimes e_{12} + \xi e_{13} \otimes e_{23} + \xi^2 e_{12} \otimes e_{13}\} + \epsilon^2 \xi e_{13} \otimes e_{12}, \end{aligned}$$

with $\xi = (\lambda/\mu)^{1/3}$. The un-deformed piece is

$$r = \phi + \sum_{i<j} e_{ij} \otimes e_{ji} = \frac{1}{3}\big(\sum e_{ii} \otimes e_{ii} - e_{11} \otimes e_{22} - e_{22} \otimes e_{33} - e_{33} \otimes e_{11}\big) + \sum_{i<j} e_{ij} \otimes e_{ji},$$

φ being completely fixed by the relations (5.7). This is in agreement with [BD], after transposition and setting $\xi = e^{u/3}, \epsilon = 1$. .

7. Elliptic r-matrices.

Here we consider the exceptional case (Proposition 6.1) in which $\hat{\Gamma}_1$ contains all the generators of $\mathcal{A}'^+$, $\mathcal{A}'$ is of type $A_{N-1}^{(1)}$ and $\tau^N = 1$.

The expression (5.25) for F_1 can be justified as before and the sum is convergent if we interpret ϵ in $\mathbb{C}$ and stipulate that

$$|\epsilon| < 1,$$

namely

$$F_1 = \frac{-1}{1-\epsilon^N} \sum_{m=1}^{N} \sum_{\sigma \in \hat{\Gamma}_1} \epsilon^m f_\sigma \otimes f_{-\tau^m \sigma}. \tag{7.1}$$

Most, but not all, of the infinite sums that arise can be made meaningful in this way. In particular, (5.25) becomes

$$(1-\epsilon^N)(1 \otimes K_\rho \partial_\rho) F_n + \sum_{m=1}^{N} \epsilon^m \left[1 \otimes f_{\tau^{-m}\rho}, F_n\right] + \sum_{m=1}^{N} \epsilon^{m-1} (f_{\tau^{-m}\rho} \otimes K^{\tau^{-m}\rho}) F_{n-1}. \tag{7.2}$$

We verify directly that it holds for $n = 1$. The second term is

$$\begin{aligned} &\frac{-1}{1-\epsilon^N} \sum_{n=1}^{N} \sum_{m=1}^{N} \epsilon^{m+n} f_{\tau^{-m-n}\rho} \otimes (K_{\tau^{-m}\rho} - K_{\tau^{1-m}\rho}) \\ &\qquad = \frac{-1}{1-\epsilon^N} \sum_{M=1}^{N} \epsilon^M f_{\tau^{-M}\rho} \otimes (K^{\tau^{-M}\rho} - K_\rho)(1-\epsilon^N). \end{aligned}$$

The last factor comes from the fact that the equation $\tau^{m+n} = \tau^M$, n, M given, only determines m Mod N. The term $K^{\sigma'}(K_\rho)$ comes from the ends of the summation while all the other terms cancel pairwise since $K^\sigma = K_{\tau\sigma}$.

The infinite product

$$F = F^1 F^2 \ldots \tag{7.3}$$

cannot be given anything more than a formal significance in the structural context but, as will be shown below, in a finite dimensional representation the question of convergence (with ϵ in $\mathbb{C}$) is not difficult. We define F^m by the (always uniquely integrable) relation (6.10),

$$(1 \otimes K_\rho \partial_\rho) F^m = -\epsilon^m (f_{\tau^{-m}\rho} \otimes K_\rho) F^m, \quad F^m = 1 - \epsilon^m \sum_\sigma f_\sigma \otimes f_{-\tau^m \sigma} + o(\epsilon^{2m}), \tag{7.4}$$

or its equivalent

$$[1\otimes f_\sigma, F^m] = -\epsilon^m\Big((f_{\tau^{-m}\sigma}\otimes K_\sigma)F^m - F^m(f_{\tau^{-m}\sigma}\otimes K^\sigma)\Big), \tag{7.5}$$

with the same initial condition. We verify that, with this definition of F^m, (7.3) satisfies (7.2) or

$$(1-\epsilon^N)(1\otimes K_\rho\partial_\rho)F + \sum_{\tau^n\sigma=\rho}\epsilon^n[1\otimes f_\sigma, F] + \sum_{\tau^n\sigma=\rho}\epsilon^n(f_\sigma\otimes K^\sigma)F = 0. \tag{7.6}$$

The range of the summation is $n = 1, 2, \dots, N$, $\sigma\in\hat\Gamma_1$. One has

$$\sum_n \epsilon^n[1\otimes f_{-\tau^{-n}\rho}, F^mF^{m+1}] = -\sum_n \epsilon^{m+n} f_{\tau^{-m-n}\rho}\otimes K_{\tau^{-n}\rho}F^mF^{m+1}$$
$$- F^m\Big\{-\sum_n \epsilon^{m+n}f_{\tau^{-m-n}\rho}\otimes K^{\tau^{-n}\rho} + \sum_n \epsilon^{m+n+1}f_{\tau^{-m-n-1}\rho}\otimes K^{\tau^{-n-1}\rho}\Big\}F^{m+1} + \dots .$$

In the second line everything cancels except for the first and the last terms, leaving

$$-\sum_n \epsilon^{m+n}(f_{\tau^{-m-n}\rho}\otimes K_{\tau^{-n}\rho})F^mF^{m+1} + (1-\epsilon^N)F^m\epsilon^{m+1}(f_{\tau^{-m-1}\rho}\otimes K_\rho)F^{m+1} + \dots .$$

The total contribution of the commutator in (7.6) is thus

$$-\sum_{n=1}^N \epsilon^{n+1}(f_{\tau^{-n-1}\rho}\otimes K_{\tau^{-n}\rho})F + (1-\epsilon^N)\sum_{m=1}^\infty F^1\dots F^m\epsilon^{m+1}(f_{\tau^{-m-1}\rho}\otimes K_\rho)F^{m+1}\dots .$$

Adding the first term in (7.6) leaves us with

$$-\sum_n \epsilon^{n+1}(f_{-\tau^{-n-1}\rho}\otimes K_{\tau^{-n}\rho})F - \epsilon(1-\epsilon^N)(f_{-\tau^{-1}\rho}\otimes K_\rho)F = -\sum_n \epsilon^n(f_{-\tau^{-n}\rho}\otimes K^{\tau^n\rho})F,$$

which is cancelled by the last term.

In the classical limit $F^m = 1+\hbar X^m + o(\hbar^2)$ and X^m satisfies (6.12). We shall solve these relations in the case of the simplest affine Kac-Moody algebra. Set

$$[f_1, f_{-1}] = (\varphi+\varphi^t)(1) = \sigma_3,$$

and

$$X^m = A^m\sigma_3\otimes\sigma_3 + B^m(f_1\otimes f_{-1} + f_0\otimes f_{-0}) + C^m(f_1\otimes f_{-0} + f_0\otimes f_{-1})$$

and impose (6.12). The result is, with $x = \sqrt{\lambda/\mu}$,

$$A^m = \sum_{n=1}^{\infty} (-\epsilon^{2n})^m x^{-n},$$

$$B^{2m} = \sum_{n=1}^{\infty} (\epsilon^{2n-1})^{2m} x^{1-n}, \quad B^{2m-1} = 0,$$

$$C^{2m-1} = \sum_{n=1}^{\infty} (\epsilon^{2n-1})^{2m-1} x^{1-n}, \quad C^{2m} = 0.$$

The deformed r-matrix is $r_\epsilon = r + X - X^t$, with

$$X = \sum_{n=1}^{\infty} X^m = \sum_n \frac{-\epsilon^{2n}}{1+\epsilon^{2n}} x^{-n} \sigma_3 \otimes \sigma_3$$
$$+ \sum_{n=1}^{\infty} \frac{\epsilon^{4n-2}}{1-\epsilon^{4n-2}} x^{1-n} (f_1 \otimes f_{-1} + f_0 \otimes f_{-0}) + \sum_{n=1}^{\infty} \frac{\epsilon^{2n-1}}{1-\epsilon^{4n-2}} x^{1-n} (f_1 \otimes f_{-0} + f_0 \otimes f_{-1}).$$

Setting $\lambda/\mu = e^{2\pi i u}$ one gets

$$(i/2)(X - X^t) = \sum_{n=1}^{\infty} \Big\{ \frac{-\epsilon^{2n}}{1+\epsilon^{2n}} (\sigma_3 \otimes \sigma_3) \sin 2n\pi u$$
$$+ \frac{\epsilon^{4n-2}}{1-\epsilon^{4n-2}} \big(x\, f_1 \otimes f_{-1} + \frac{1}{x}\, f_{-1} \otimes f_1 \big) \sin(2n-1)\pi u$$
$$+ \frac{\epsilon^{2n-1}}{1-\epsilon^{4n-2}} \big(\sqrt{1/\mu\lambda}\, f_1 \otimes f_1 + \sqrt{\mu\lambda}\, f_{-1} \otimes f_{-1} \big) \sin(2n-1)\pi u \Big\}.$$

The trigonometric r-matrix (2.15) is

$$\frac{i}{2} \bigg(\frac{1}{\tan \pi u} (\sigma_3 \otimes \sigma_3) + \frac{1}{\sin \pi u} \big(\sqrt{x}\, f_1 \otimes f_{-1} + \sqrt{1/x}\, f_{-1} \otimes f_1 \big) \bigg).$$

Adding, one finds the series expansion of elliptic functions, and complete agreement with the elliptic r-matrices of [BD]. To transform to their notation replace

$$f_1 \to \sqrt{\lambda}\, e_{12},\ f_{-1} \to \sqrt{1/\lambda}\, e_{21} \tag{7.7}$$

Finally, we shall show that the expression for the Universal Elliptic R-matrix as an infinite product is both meaningful and useable, by projecting on a finite dimensional represention. We limit ourselves to the fundamental representation of $sl(2)$. After rescaling of the generators as in (7.7), F^m and R_ϵ take the form

$$F^m = \begin{pmatrix} a^m & & & d^m \\ & b^m & c^m & \\ & c^m & b^m & \\ a^m & & & d^m \end{pmatrix}, \quad R_\epsilon = \begin{pmatrix} a & & & d \\ & b & c & \\ & c & b & \\ a & & & d \end{pmatrix}.$$

The matrix elements are completely determined by the recursion relation (7.5); namely for $m = 1, 2, \ldots$,

$$a^{2m-1} = 1 - \epsilon^{4m-2}, \quad b^{2m-1} = 1 - \epsilon^{4m-2}\frac{q^2}{x}, \quad c^{2m-1} = 0, \quad d^{2m-1} = \epsilon^{2m-1}(\frac{1}{q} - q)\sqrt{\frac{1}{x}},$$
$$a^{2m} = 1 - \epsilon^{4m}\frac{q^2}{x}, \qquad b^{2m} = 1 - \epsilon^{4m}\frac{1}{x}, \qquad c^{2m} = \epsilon^{2m}\sqrt{\frac{1}{x}}(\frac{1}{q} - q), \qquad d^{2m} = 0,$$

and

$$a + d \ : \ a - d \ : \ b + c \ : \ b - c = \frac{dn(u+\rho)}{dn(u-\rho)} : 1 : \frac{cn(u+\rho)}{cn(u-\rho)} : \frac{sn(u+\rho)}{sn(u-\rho)}.$$

Acknowledgements. I thank Moshe Flato, Kajiwara and Tetsuji Miwa for discussions, and H. Araki for generous hospitality at the Research Institute for Mathematical Sciences, Kyoto University. I thank the Ministry of Education, Science, Sports and Culture for financial support.

References.

[BD] A.A. Belavin and V.G. Drinfeld, Sov.Sci.Rev.Math.**4** (1984) 93-165.

[F] C. Frønsdal, "Generalization and Deformations of Quantum Groups; Quantization of all simple Lie Bi-algebras." q-alg 951000

[J] M. Jimbo, Commun.Math.Phys.

[K] V.G. Kac, *Infinite Dimensional Lie Algebras*, Cambridge University Press 1990.

[L] G. Lusztig, *Quantum Groups* 1993.

[R] N.Yu. Reshetikhin, Lett.Math.Phys. **20** (1990) 331-336.